TRANSLATION OF RUSSIAN PAPERS ON THE ERRORS IN NUMERICAL INTEGRATION OF THE EQUATIONS OF CELESTIAL MECHANICS

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The Estimation of the Error Resulting from Numerical Integration of the Equations of Celestial Mechanics
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and

The Accumulation of Numerical Integration Errors in Some Problems of Celestial Mechanics
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FOREWORD

The following articles were translated because they are relevant to computations performed for the Navy Space Surveillance Project and the Transit Navigation Satellite Project. The articles should also be of interest to others concerned with the subject of errors in numerical integration.

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THE ESTIMATION OF THE ERROR RESULTING FROM NUMERICAL INTEGRATION OF THE EQUATIONS OF CELESTIAL MECHANICS

V. F. Myachin

This article presents the result of the application of a general theory of estimation of the numerical integration error of differential equations, proposed by Professor S. M. Lozinsky, to equations of undisturbed motion of celestial mechanics. In applying the theory, consideration was given to the random character of the round-off error. Here, in particular, it is shown that the new theory gives qualitative confirmation of the well-known result of Brouwer, which asserts that after a sufficiently large number, k, of integration steps the error in the coordinates of elliptic motion has the order of growth $k^{\frac{3}{2}}$.

Introduction

Let us consider a system of equations of disturbed motion:

$$\begin{aligned} \ddot{x} &= -K^2 \frac{x}{r^3} + R_x, \\ \ddot{y} &= -K^2 \frac{y}{r^3} + R_y, \\ \ddot{z} &= -K^2 \frac{z}{r^3} + R_z, \\ r^2 &= x^2 + y^2 + z^2. \end{aligned} \qquad \text{(A)}$$

As is generally known, the most prevalent method of approximating the solution of such equations is by numerical integration; in addition it is known that the application of any formula of numerical integration inevitably leads to an accumulation of error in the computed approximate solution, which arises in two ways:

1) use in the integration formulas of a finite number of terms;

2) errors in initial conditions and round-off in each step of integration.

At the present time we do not have available strict, practically acceptable estimates of the error of numerical integration of differential equations; however, there exist so-called rough estimates, which were obtained by rough reasoning.

Thus, Newcomb (1898) regarded the error in the numerical integration of any differential equation as the result of the sum of the round-off errors at each step (by analogy with the error of approximate computation of a definite integral); counting these errors as independent random variables (in the sense of the theory of probability), he came to the conclusion that the accumulation of error after k steps of numerical integration is of the order of $k^{\frac{3}{2}}$ for equations of the second order and $k^{\frac{1}{2}}$ for equations of the first order.

Applying this assertion to the estimation of the accumulation of round-off errors in the method of special perturbations, one can draw a conclusion about the advantage of obtaining the perturbations in the elements compared with their computation by the method of Cowell or Encke (i.e., in the coordinates). In the first case the perturbations are obtained as a result of single integration for all elements except the mean longitude, which requires double integration, and, consequently, only in this element is there a serious accumulation of error. If, however, the perturbations are computed by the method of Cowell or Encke, then for every coordinate there will be an equation of the second order and the expected error in each coordinate will be of the order of $k^{\frac{3}{2}}$.

Brouwer (1937), however, showed that this conclusion is incorrect and that for elliptic motion the accumulation of errors in the coordinates is equivalent to an error in mean longitude proportional to $k^{\frac{3}{2}}$, and to errors in the other elements proportional to $k^{\frac{1}{2}}$.

Consequently, for circular motion, where the mean motion is a constant quantity, the error in the coordinates by the integration method of Cowell or Encke is proportional to $k^{\frac{1}{2}}$. In this connection Brouwer first noticed the essential difference between the approximate computation of a definite integral and the numerical integration of differential equations; in the latter case the computation of each tabular value uses the result of a prior integration, so that the error in this case cannot possibly be considered as the result of simple summation of errors at each step.

Still, because of loose reasoning, the result of Brouwer on the quantitative side appears rough: in particular, it does not reflect the experimentally known fact that the error has an oscillatory character; moreover, at a sufficiently large number of steps of integration the formulas give an underestimate, i.e., the true error exceeds prediction.

At a later time Professor S. M. Lozinsky proposed a number of general theorems for estimating the error in numerical integration of systems of differential equations.

The present paper presents the result of the application to System (A) of Lozinsky's so-called "linearized estimate"[1], which was somewhat modified and adapted in the interest of our problem. It is shown that if, following Newcomb and Brouwer, one considers the round-off errors as independent random variables, then for the accumulation of error in the coordinates of elliptic motion the indicated estimate gives the order of $k^{\frac{3}{2}}$, and for circular motion the order of $k^{\frac{1}{2}}$.

[1]At the present time this estimate is in press. By permission of the author we borrowed it from a course of lectures "Approximate Solution of Differential Equations", read by S. M. Lozinsky at Leningrad University in 1955-1957.

This method permits one to take account not only of the influence of round-off errors on the error in the coordinates, but also the effects produced by errors in the initial conditions and by the discard of the remainder term in the integration formulas. Moreover, it is possible to keep track of the variation of the error separately in each coordinate.

The exposition is based upon an example of Cowell's method.

1. A FEW GENERAL STATEMENTS ABOUT MATRICES AND SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

Let there be given the rectangular matrix:

$$A = \begin{Vmatrix} a_{11}, & a_{12}, & \ldots, & a_{1n} \\ a_{21}, & a_{22}, & \ldots, & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1}, & a_{m2}, & \ldots, & a_{mn} \end{Vmatrix}.$$

We shall use the following notation:

$$a_{ij} \equiv (A)_{ij}; \quad A \equiv \| a_{ij} \|.$$

A matrix consisting of one column is called a vector. Let $\mathfrak{x}$ be a vector with components $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$.

We shall denote:

$$\mathfrak{x} \equiv (x^{(1)}, x^{(2)}, \ldots, x^{(n)}); \quad x^{(i)} \equiv (\mathfrak{x})^{(i)}.$$

By $|\mathfrak{x}|$ we shall denote a vector, defined by the equation

$$(|\mathfrak{x}|)^{(i)} = |(\mathfrak{x})^{(i)}|.$$

Finally, we introduce the norm of the vector $\mathfrak{x}$, denoted by $\|\mathfrak{x}\|$ and defined by the equation

$$\|\mathfrak{x}\| \equiv \max_{i=1}^{n} |(\mathfrak{x})^{(i)}|.$$

A vector will be considered as a special case of a matrix.

For a given matrix A we denote by $\tilde{A}$ the matrix defined by the equation

$$(\tilde{A})_{ij} \equiv (A)_{ji};$$

it is called the transpose.

The transpose of a vector is a matrix consisting of one row:

$$\tilde{x} = |x^{(1)}, x^{(2)}, \ldots, x^{(n)}|.$$

A matrix having the same number of rows and columns is called a square matrix. The elements $(A)_{ii}$ of a square matrix A form the principal diagonal.

Operations upon matrices:

1. Addition of two matrices. By definition:

$$(A+B)_{ij} \equiv (A)_{ij} + (B)_{ij}.$$

2. Multiplication of a matrix A by a number λ. By definition:

$$(\lambda A)_{ij} \equiv \lambda (A)_{ij}.$$

3. Matrix multiplication. Let a matrix A have n columns, and a matrix B have n rows; then by definition:

$$(AB)_{ij} \equiv \sum_{m=1}^{n} (A)_{im} (B)_{mj}.$$

If the number of columns of A is not equal to the number of rows of B, then the product AB is not defined. Matrix multiplication possesses the associative property, and multiplication and addition possess the distributive property:

$$(AB)C = A(BC),$$
$$C(A+B)D = CAD + CBD,$$

if all indicated products are defined.

The operation of matrix multiplication, generally speaking, does not have the commutative property:

$$AB \neq BA.$$

For example, for the vector $\bar{x} \equiv (x^{(1)}, x^{(2)}, \ldots, x^{(n)})$ and its transpose $\tilde{\bar{x}}$ we have:

$$\tilde{\bar{x}}\bar{x} = x^{(1)^2} + x^{(2)^2} + \ldots + x^{(n)^2},$$

but

$$\bar{x}\tilde{\bar{x}} = \begin{vmatrix} x^{(1)^2}, & x^{(1)}x^{(2)}, & \ldots, & x^{(1)}x^{(n)} \\ x^{(2)}x^{(1)}, & x^{(2)^2}, & \ldots, & x^{(2)}x^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ x^{(n)}x^{(1)}, & x^{(n)}x^{(2)}, & \ldots, & x^{(n)^2} \end{vmatrix}. \quad (1)$$

4. The transpose of the product of two matrices:

$$(\widetilde{AB}) = \tilde{B}\tilde{A}. \quad (2)$$

5. Matrix differentiation. Let $A(t)$ be a matrix representing a function of the scalar argument t; then by definition:

$$\{\dot{A}(t)\}_{ij} = \frac{d}{dt}\{A(t)\}_{ij}.$$

6. Integration. By definition:

$$\left\{\int_{t_0}^{t} A(t')\,dt'\right\}_{ij} = \int_{t_0}^{t} \{A(t')\}_{ij}\,dt'.$$

Henceforth, we will speak only of square matrices. A matrix whose elements are all zero is called null and is denoted by **0**. A matrix whose elements on the principal diagonal are equal to unity, and all the rest are zero, is called a unit matrix and denoted by I.

$$I = \begin{vmatrix} 1, & 0, & \ldots, & 0 \\ 0, & 1, & \ldots, & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & \ldots, & 1 \end{vmatrix}.$$

For any matrix A it is true that

$$AI = IA = A.$$

The matrix S, possessing the property

$$\tilde{S}S = I,$$

is called orthogonal.

Consider now the system of linear differential equations

$$\dot{y}^{(i)} = \sum_{j=1}^{n} p_{ij}(t)\, y^{(j)}.$$

Introducing the vector

$$y = \{y^{(1)}, y^{(2)}, \ldots, y^{(n)}\}$$

and the matrix

$$P(t) = \| p_{ij}(t) \|,$$

we write this system in the form

$$\dot{y} = P(t)\, y. \tag{3}$$

We introduce the matrices of order n $U(t_0, t)$, $V(t_0, t)$, defined by the conditions:

$$\begin{aligned} \ddot{U} &= P(t)\, U, \quad U(t_0, t_0) = I, \quad \dot{U}(t_0, t_0) = 0 \\ \ddot{V} &= P(t)\, V, \quad V(t_0, t_0) = 0, \quad \dot{V}(t_0, t_0) = I, \end{aligned} \tag{4}$$

which are called respectively the first and second fundamental solutions of System (3). The columns of these matrices represent the $2n$ linear independent solutions of System (3).

Knowing the fundamental solutions of System (3), one can solve the nonhomogeneous problem:

$$\ddot{z} = P(t)\, z + \tau(t), \quad z(t_0) = z_0, \quad \dot{z}(t_0) = \dot{z}_0.$$

Its solution has the form

$$z(t) = U(t_0,\ t)\, z_0 + V(t_0,\ t)\, \dot{z}_0 + \int_{t_0}^{t} V(\xi,\ t)\, \tau(\xi)\, d\xi. \qquad (5)$$

There occurs the following dependence between the fundamental solutions:

$$V(t_0,\ t) = \int_{t_0}^{t} U(\xi,\ t)\, d\xi$$

or

$$U(t_0,\ t) = -\frac{\partial}{\partial t_0} V(t_0,\ t). \qquad (6)$$

For the proof of these relations it is sufficient that the expression in the right member of the first equation satisfies the second group of conditions (4).

2. THE REMAINDER TERM OF COWELL'S METHOD

Let us consider the vector function

$$\bar{G}(z) \equiv (G^{(1)}(z),\ G^{(2)}(z),\ \ldots,\ G^{(n)}(z))$$

of the scalar argument z and introduce for it the following differences:

$$\bar{G}^{(1)}_{\frac{1}{2}} \equiv \bar{G}(1) - \bar{G}(0), \quad \bar{G}^{(2)}_0 \equiv \bar{G}^{(1)}_{\frac{1}{2}} - \bar{G}^{(1)}_{-\frac{1}{2}}, \quad \bar{G}^{(3)}_{\frac{1}{2}} \equiv \bar{G}^{(2)}_1 - \bar{G}^{(2)}_0, \ \ldots;$$
$$\bar{G}^{(j)}_0 \equiv \frac{1}{2}\left(\bar{G}^{(j)}_{-\frac{1}{2}} + \bar{G}^{(j)}_{\frac{1}{2}}\right) \qquad (j = 1,\ 3,\ 5 \ldots). \qquad (7)$$

We state the interpolation formula of Sterling

$$\bar{G}(z)=\bar{G}(0)+\sum_{j=1}^{2\nu+1} A_j(z)\,\bar{G}_0^{(j)}+\bar{R}, \tag{8}$$

where

$$A_{2j}(z)\equiv\frac{z^2(z^2-1)(z^2-4)\ldots[z^2-(j-1)^2]}{(2j)!},$$

$$A_{2j-1}(z)\equiv 2j\,\frac{A_{2j}(z)}{z} \qquad (j=1,\ 2,\ \ldots).$$

If the function $\bar{G}(z)$ has a derivative of the order $2\nu+3$, then the remainder term $\bar{R}$ can be written in the form

$$\bar{R}=A_{2\nu+2}\,\frac{d^{2\nu+2}\bar{G}(\zeta_1(z))}{dz^{2\nu+2}}+A_{2\nu+3}\,\frac{d^{2\nu+3}\bar{G}(\zeta_2(z))}{dz^{2\nu+3}}, \qquad |\zeta_i|\leqslant\nu+1. \tag{9}$$

Remark. For each scalar function entered into the expression $\bar{G}(z)$, the quantities ζ_i take, generally speaking, different values.

We consider now the system of differential equations:

$$\ddot{x}^{(i)}=F^{(i)}(t,\ x^{(1)},\ \ldots,\ x^{(n)}) \quad (i=1,\ 2,\ \ldots\ n).$$

Introducing the vectors

$$\bar{x}\equiv(x^{(1)},\ x^{(2)},\ \ldots,\ x^{(n)}), \quad \bar{F}\equiv(F^{(1)},\ F^{(2)},\ \ldots,\ F^{(n)}),$$

we write the system in the form

$$\ddot{\bar{x}}=\bar{F}(t,\ \bar{x}). \tag{10}$$

We denote by $\bar{x}(t)$ a solution of System (10) and by h the step of the numerical integration, and we put for brevity

$$\bar{F}(t,\ \bar{x}(t))\equiv\bar{f}(t), \quad t_0+kh\equiv t_k,$$

$$\bar{x}(t_k)\equiv\bar{x}_k, \quad \bar{x}_{k+1}-2\bar{x}_k+\bar{x}_{k-1}\equiv\Delta^2\bar{x}_k.$$

Then successive integration of System (10) gives

$$\dot{x}(t') - \dot{x}_k = \int_{t_k}^{t'} f(t'')\,dt'',$$

$$x_{k+1} - x_k - h\dot{x}_k = \int_{t_k}^{t_{k+1}} dt' \int_{t_k}^{t'} f(t'')\,dt'' = \int_{t_k}^{t_{k+1}} (t_{k+1} - t') f(t')\,dt' = h^2 \int_0^1 (1 - z) f(t_k + hz)\,dz. \tag{11}$$

Writing an analogous relation for the points t_{k-1}, t_k and combining this with Equation (11), we obtain finally

$$\Delta^2 x_k = h^2 \int_0^1 (1 - z)\, \bar{G}(z)\,dz, \tag{12}$$

where it is assumed

$$\bar{G}(z) \equiv f(t_k + hz) + f(t_k - hz). \tag{13}$$

We introduce now for the function $f(t)$ the differences, analogous to (7):

$$f_k \equiv f(t_k),$$
$$f^{(1)}_{k+\frac{1}{2}} \equiv f_{k+1} - f_k$$
$$f^{(2)}_k \equiv f^{(1)}_{k+\frac{1}{2}} - f^{(1)}_{k-\frac{1}{2}}$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$f^{(j)}_k = \frac{1}{2}\left(f^{(j)}_{k-\frac{1}{2}} + f^{(j)}_{k+\frac{1}{2}}\right) \qquad (j = 1,\ 3,\ 5\ \ldots)$$

Then, noticing that, according to (13),

$$\bar{G}_0^{(2j)} = 2 f_k^{(2j)}, \quad \bar{G}_0^{(2j-1)} = 0 \qquad (j = 1,\ 2,\ \ldots),$$

and substituting (8) in (12), we obtain

$$\Delta^2 x_k = h^2 \left[f_k + \sum_{j=1}^{\nu} \alpha_{2j} f_k^{(2j)} + q_k \right], \tag{14}$$

where

$$\alpha_j \equiv 2\int_0^1 (1-z)A_j(z)\,dz \qquad (j=1,\ 2,\ldots). \tag{15}$$

If the function $f(t)$ has a continuous derivative of order $2\nu+3$, ı, according to (9), the remainder term $\bar{q}_k$ can be presented in the form

$$\bar{q}_k = \frac{1}{2}\alpha_{2\nu+2}\frac{d^{2\nu+2}\bar{G}(\zeta_1)}{dz^{2\nu+2}} + \frac{1}{2}\alpha_{2\nu+3}\frac{d^{2\nu+3}\bar{G}(\zeta_2)}{dz^{2\nu+3}}, \qquad |\zeta_i| \leqslant \nu+1. \tag{16}$$

To be convinced of this, we introduce the numbers m_i, M_i, $i=1,\ 2,\ \ldots\ n$, representing respectively the minimum and maximum $(2\nu+3)$ th derivative of the function $G^{(i)}(z)$ in the interval $[-\nu-1,\ \nu+1]$:

$$m_i \leqslant \frac{d^{2\nu+3}G^{(i)}(\zeta_2(z))}{dz^{2\nu+3}} \leqslant M_i.$$

Further, noticing that for $0 \leqslant z \leqslant 1$ the factor $(1-z)A_{2\nu+3}(z)$ represents a quantity of constant sign (we say, positive), we multiply it by the inequality and integrate

$$m_i \leqslant \frac{\int_0^1 (1-z)A_{2\nu+3}(z)\frac{d^{2\nu+3}G^{(i)}(\zeta_2(z))}{dz^{2\nu+3}}\cdot dz}{\frac{1}{2}\alpha_{2\nu+3}} \leqslant M_i.$$

Hence, by the known property of continuous functions, we prove the existence of a point ζ in the interval $[-\nu-1,\ \nu+1]$, such that

$$\frac{\int_0^1 (1-z)A_{2\nu+3}(z)\frac{d^{2\nu+3}G^{(i)}(\zeta_2(z))}{dz^{2\nu+3}}\cdot dz}{\frac{1}{2}\alpha_{2\nu+3}} = \frac{d^{2\nu+3}G^{(i)}(\zeta)}{dz^{2\nu+3}}.$$

Integration of the first term from (9) is carried out analogously.

For sufficiently small h expression (16) can be transformed to the form

$$\bar{q}_k \approx \alpha_{2\nu+2} h^{2\nu+2} \frac{d^{2\nu+2} f(t_k)}{dt^{2\nu+2}} + \alpha_{2\nu+3} \zeta_2 h^{2\nu+4} \frac{d^{2\nu+4} f(t_k)}{dt^{2\nu+4}}, \qquad |\zeta_2| \leqslant \nu+1. \tag{16'}$$

Equation (14) presents Cowell's method in the difference form. Corresponding to it is the following sum method

$$x_k = h^2 \left[f_k^{(-2)} + \alpha_2 f_k + \sum_{j=2}^{\nu} \alpha_{2j} f_k^{(2j-2)} + \bar{q}'_k \right], \tag{17}$$

where

$$\Delta^2 f_k^{(-2)} \equiv f_k, \quad \Delta^2 \bar{q}'_k \equiv \bar{q}_k.$$

We cite a few values of the coefficients (15):

α_2	$\frac{1}{12}$	α_3	$-\frac{7}{180}$
α_4	$-\frac{1}{240}$	α_5	$\frac{37}{5040}$
α_6	$\frac{31}{60480}$	α_7	$-\frac{199}{129600}$
α_8	$-\frac{289}{3628800}$	α_9	$\frac{1}{3024}$

3. ROUND-OFF ERRORS AND THE ERROR IN COWELL'S METHOD

If in the precise formulas (14), (17) the remainder terms $\bar{q}_k$, $\bar{q}'_k$, are omitted, if for all x_k their approximate values are substituted, and if the necessary round-off is effected, then we obtain Cowell's difference method and sum method, respectively.

We shall assume that System (10) with the initial conditions

$$x(t_0) = x_0, \quad x(t_{-1}) = x_{-1} \tag{18}$$

is integrated by the sum method with the step h (within the domain where the sought for solution exists), and as a result of the integration we obtain the table of values

$$\ldots, X_{-2}, X_{-1}, X_0, X_1, \ldots, X_k, \ldots, \tag{19}$$

at the corresponding times

$$\ldots, t_{-2}, t_{-1}, t_0, t_1, \ldots, t_k, \ldots,$$

so that

$$X_k \approx x(t_k) \equiv x_k.$$

We introduce the quantity τ_k, defined by the equation

$$\tau_k \equiv \Delta^2 X_k - \Phi_k - \sum_{j=1}^{\nu} \alpha_{2j}^{\nu} \Phi_k^{(2j)} \qquad (k = 0, 1, \ldots), \tag{20}$$

where it is assumed

$$\Phi(t, x) \equiv h^2 F(t, x), \; \Phi_k \equiv \Phi(t_k X_k), \quad \Phi_k^{(2)} \equiv \Delta^2 \Phi_k, \text{ etc.}$$

The quantity τ_k is called the error of the sum method.

If System (10) is integrated by the difference method, then the quantity τ_k, which is defined by Equation (20), is called the error of the difference method.

The error arises mainly due to round-off errors at each step of the numerical integration.

For investigation of the question of calculation of the error we limit ourselves to a very simple case of Formula (17):

$$x_k = h^2 \left[f_k^{(-2)} + \frac{1}{12} f_k + q_k' \right]. \tag{21}$$

If one digresses from round-off errors, then the computations by the method, corresponding to the precise equation (21), can be presented in the form of the following algorithm.

1. The approximation of the values of the initial conditions (18) X_0, X_{-1} are given and we enter them in the table of values (19); we compute Φ_0, Φ_{-1}, and also the quantities

$$\Phi_0^{(-2)} \equiv X_0 - \frac{1}{12}\Phi_0,$$

$$\Phi_{-\frac{1}{2}}^{(-1)} \equiv X_0 - X_{-1} - \frac{1}{12}(\Phi_0 - \Phi_{-1}).$$

2. From the known Φ_0, Φ_{-1} we extrapolate (we predict) the value of $\Phi_{1,0}$; we compute $\Phi_1^{(-2)}$ by the formula

$$\Phi_1^{(-2)} \equiv \Phi_0^{(-2)} + \Phi_{-\frac{1}{2}}^{(-1)} + \Phi_0$$

and we assume

$$X_{1,0} \equiv \Phi_1^{(-2)} + \frac{1}{12}\Phi_{1,0}.$$

3. We compute

$$\Phi(t_1, X_{1,0}) \equiv \Phi_{1,1}$$

and we assume

$$X_{1,1} \equiv \Phi_1^{(-2)} + \frac{1}{12}\Phi_{1,1}.$$

4. We compute

$$\Phi(t_1, X_{1,1}) \equiv \Phi_{1,2}$$

and we form the difference

$$\Phi_{1,1} - \Phi_{1,2} \equiv \lambda_1;$$

if it is found that

$$\|\lambda_1\| \leqslant \lambda,$$

where λ is some preassigned number, then we finish this iteration process at the first step of integration and enter the value of $X_{1,1}$ in the table of values (19) in place of X_1. If, however, the above inequality is not satisfied, then, repeating the operations analogous to those set forth in 3, we construct the following approximations:

$$X_{1,2},\ X_{1,3},\ \ldots,\ X_{1,l}$$

until the inequality is satisfied

$$\|\Phi(t_1,\ X_{1,l-1})-\Phi(t_1,\ X_{1,l})\|\equiv\|\lambda_1\|\leqslant\lambda;$$

then we enter the vector $X_{1,l}$ in the table of values (19) in place of X_1

5. If in the table of values (19) is already written all X_j up to $j=k-1$ inclusive, then for computing X_k we find the value of $\Phi_k^{(-2)}$, defined by the equations

$$\Phi_k^{(-2)}\equiv\Phi_0^{(-2)}+k\Phi_{-\frac{1}{2}}^{(-1)}+\sum_{j=0}^{k-1}(k-j)\Phi_j \tag{22}$$

or

$$\Delta^2\Phi_{k-1}^{(-2)}\equiv\Phi_{k-1}, \tag{22'}$$

we extrapolste the value of $\Phi_{k,0}$ and construct the following approximations by the formulas

$$\begin{aligned}X_{k,l}&\equiv\Phi_k^{(-2)}+\frac{1}{12}\Phi_{k,l},\\ \Phi_{k,l+1}&\equiv\Phi(t_k,\ X_{k,l}),\quad l=0,\ 1,\ \ldots\end{aligned} \tag{23}$$

The iterations are continued until the inequality

$$\|\Phi_{k,l}-\Phi_{k,l+1}\|\equiv\|\lambda_k\|\leqslant\lambda. \tag{23'}$$

is satisfied.

Thus, if all X_k from (19) are obtained in l approximations on the kth step, that is if

$$X_k = X_{k,l} \qquad (k = 1, 2, \ldots),$$

then it is necessary for the condition (23') to be fulfilled.

The quantity $\bar{\lambda}_k$ is called the error of method (21).

Let us suppose now, that all $\Phi_{k,l}$, Φ_k are computed with the errors $\rho_{k,l}$, ρ_k, subject to the inequalities

$$\begin{aligned} \|\bar{\rho}_{k,l}\| &\leqslant \rho \qquad (k \geqslant 1,\ l \geqslant 1), \\ \|\bar{\rho}_k\| &\leqslant \rho \qquad (k \geqslant -1), \end{aligned}$$

where ρ is a fixed number. This generates in the quantities $\Phi_k^{(-2)}$ the errors $\bar{\rho}'_k$, which, according to (22) and (22'), satisfy the relations

$$\Delta^2 \bar{\rho}'_k = \bar{\rho}_k \qquad (k \geqslant 2). \tag{24}$$

Let us suppose further, that in the computation of $X_{k,l}$, X_k round-off errors $\mu_{k,l}$, $\bar{\mu}_k$, are introduced, satisfying the inequalities

$$\begin{aligned} \|\bar{\mu}_{k,l}\| &\leqslant \mu \qquad (k \geqslant 1,\ l \geqslant 1), \\ \|\bar{\mu}_k\| &\leqslant \mu \qquad (k \geqslant 1), \end{aligned}$$

where μ is a fixed number (for example, if all $X_{k,l}$, X_k are computed to s decimal places, then it follows that $\mu = \frac{1}{2} 10^{-s}$).

With regard for the errors introduced above, Equation (23) and the condition (23') must be written in the form

$$X_{k,l} = \Phi_k^{(-2)} + \rho'_k + \frac{1}{12}\Phi_{k,l} + \text{er.}\left(\frac{1}{12}\Phi_{k,l}\right) + \mu_{k,l} \ (k \geqslant 1); \tag{25}$$

[where

$$\text{er.} \left(\frac{1}{12}\Phi_{k,l}\right) = \text{error in} \left(\frac{1}{12}\Phi_{k,l}\right)]$$

$$\|\Phi_{k,l} - \Phi_{k,l+1} + \rho_{k,l} - \rho_{k,l+1}\| = \|\bar{\lambda}_k\| \leqslant \lambda \qquad (k \geqslant 1). \tag{25'}$$

Proceeding now to the calculation of the error

$$\tau_k = \Delta^2 X_k - \Phi_k - \frac{1}{12}\Delta^2\Phi_k,$$

we introduce the auxiliary quantity

$$\tau'_k \equiv X_k - \Phi_k^{(-2)} - \frac{1}{12}\Phi_k \qquad (k \geqslant -1); \tag{*}$$

then, according to (25) and (25'), we find

$$\tau'_k = \rho'_k + \text{er.}\left(\frac{1}{12}\Phi_k\right) + \bar{\mu}_k + \frac{1}{12}(\Phi_{k,l} - \Phi_{k,l+1}) = \rho'_k + \frac{1}{12}\rho_{k,l+1} + \text{er.}\left(\frac{1}{12}\right)\Phi_k +$$
$$+ \bar{\mu}_k + \frac{1}{12}\bar{\lambda}_k \qquad (k \geqslant 1).$$

Hence, observing that

$$\tau_k = \Delta^2 \tau'_k, \tag{**}$$

and applying the operator Δ^2 to both members of the above equation, with the aid of (24) we find finally

$$\tau_k = \rho_k + \frac{1}{12}\Delta^2\rho^*_k + \text{er.}\left(\frac{1}{12}\right)\Delta^2\Phi_k + \Delta^2\bar{\mu}_k + \frac{1}{12}\Delta^2\bar{\lambda}_k \qquad (k \geqslant 2), \tag{26}$$

where it is assumed

$$\rho^*_k \equiv \rho_{k,l+1}.$$

Remark. If one makes use of the definition of $\Phi_0^{(-2)}$ and introduces also

$$\Phi_{-1}^{(-2)} \equiv \Phi_0^{(-2)} - \Phi_{-\frac{1}{2}}^{(-1)},$$

then Equation (*) for $k = 0, -1$ gives

$$\tau'_0 = \tau'_{-1} = 0.$$

Hence, according to (**) it is easy to show that Formula (26) can be retained for $k = 0, 1$, if ρ^*_0, ρ^*_{-1} are replaced by the quantities ρ_0, ρ_{-1}, and likewise it is assumed that

$$\mu_0 = \mu_{-1} = 0, \quad \bar{\lambda}_0 = \bar{\lambda}_{-1} = 0$$

and in the expression er. $\left(\frac{1}{12}\right)\Delta^2\Phi_k$ it is considered that

$$\Phi_0 = \Phi_{-1} = 0.$$

If the integration of System (10) is carried out by the difference method corresponding to the exact formula

$$\Delta^2 x_k = h^2\left(f_k + \frac{1}{12}\Delta^2 f_k + q_k\right),$$

then, repeating all of the above discussion with the corresponding changes, we come to the following expression for the error ε_k:

$$\varepsilon_k = \rho_k + \frac{1}{12}\Delta^2\rho_k^* + \text{er.}\left(\frac{1}{12}\right)\Delta^2\Phi_k + \mu_k' + \frac{1}{12}\Delta^2\lambda_k \qquad (k \geqslant 0), \tag{26'}$$

where by μ_k' are meant the round-off errors for $\Delta^2 X_k$, subject to the condition

$$\|\mu_k'\| \leqslant \mu' \qquad (k \geqslant 0),$$

and all remaining quantities have the previous meaning.

Later we will make use of the expressions (26) and (26') in place of the approximate values for the error ε_k, defined by Equation (20).

Remark. The quantities ρ_k were defined as errors arising from the computation of the vectors Φ_k, if in the latter the components of the vectors X_k are considered as precise numbers. Following Newcomb and Brouwer, we will consider these errors as independent round-off errors.

4. THE LINEARIZED ERROR ESTIMATE OF COWELL'S METHOD

Let us suppose that the times $\ldots, t_{-2}, t_{-1}, t_0, t_1, \ldots, t_k, \ldots$ correspond to the quantities (19) obtained by numerical integration of System (10) with the initial conditions (18) by Cowell's method. This means that these quantities satisfy the following relations:

$$\Delta^2 X_k = h^2\left[F_k + \sum_{j=1}^{\nu} \alpha_{2j} F_k^{(2j)}\right] + \varepsilon_k \qquad (k \geqslant 0), \tag{27}$$

where it is assumed

$$F_k \equiv F(t_k, X_k), \qquad F_k^{(2)} \equiv \Delta^2 F_k, \text{ etc.}$$

and by τ_k is meant the error which is computed by Formula (26) or (26').

Designating by $x(t)$ the exact solution of System (10) with the initial conditions (18) and assuming $x(t_k) \equiv x_k$, we introduce the following definitions:

a) the error of approximation of the solution X_k (or the error of Cowell's method)

$$X_k - x_k \equiv \delta_k \equiv (\delta_k^{(1)}, \delta_k^{(2)}, \ldots, \delta_k^{(n)}); \tag{28}$$

b) the Jacobian $J(t, x)$ of System (10)

$$(J(t, x))_{ij} \equiv \frac{\partial F^{(i)}(t, x^{(1)}, x^{(2)}, \ldots, x^{(n)})}{\partial x^{(j)}}$$

(here x represents the set of the independent coordinates);

c) the Jacobian in the unknown exact solution

$$J(t) \equiv J(t, x(t));$$

d) finally, we obtain

$$Q_k \equiv \int_0^1 J(t_k, x_k + \alpha\delta_k)\, d\alpha$$

[The quantities Q_k defined only for the condition that the segments L_k, connecting points (t_k, x_k) and (t_k, X_k) of the $(n+1)$ dimensional space, are contained in the region of definition of the solution $x(t)$].

Subtracting (14) from (27) and using the obvious identity

$$F_k - f_k \equiv Q_k\delta_k,$$

we find

$$\frac{\Delta^2\delta_k}{h^2} = Q_k\delta_k + \sum_{j=1}^{\nu} a_{2j}\Delta^{2j}(Q_k\delta_k) + \frac{\tau_k}{h^2} - q_k \qquad (k \geqslant 0), \tag{29}$$

where Δ^{2j} is the central difference operator of order $2j$.

Equation (29) is called the equation of errors.

In order to simplify the problem, we replace this equation by the following approximation:

$$\frac{\Delta^2 \delta_k}{h^2} = J(t_k)\,\delta_k + \frac{\tau_k}{h^2} - \bar{q}_k; \tag{29'}$$

Equation (29') is called the linearized equation of errors (Lozinsky).

Let us introduce the fundamental solutions $U(t_0,\ t)$, $V(t_0,\ t)$ of the system $\ddot{y} = J(t)y$, defined by the conditions of type (4). Then the general solution of Equation (29') by analogy with (5) can be written approximately in the form

$$\delta_k \approx \delta_{k,\,0} + \delta_{k,\,\tau} + \delta_{k,\,q} \qquad (k \geqslant 1), \tag{30}$$

where it is assumed

$$\delta_{k,0} \equiv U(t_0,\ t_k)\,\delta_0 + \frac{V(t_0,\ t_k)}{h}(\delta_0 - \delta_{-1}), \tag{31}$$

$$\delta_{k,\,\tau} \equiv \sum_{m=0}^{k-1} V(t_m,\ t_k)\frac{\tau_m}{h}, \tag{32}$$

$$\delta_{k,\,q} \equiv -\sum_{m=0}^{k-1} V(t_m,\ t_k)\,\bar{q}_m h. \tag{33}$$

Every estimate according to the absolute value of the right member of Equation (30) is called the linearized error estimate of method (14) or (17).

The quantity $\delta_{k,0}$ represents the error created by the errors

$$\delta_0 = X_0 - x_0, \quad \delta_{-1} = X_{-1} - x_{-1}$$

in the initial conditions (18) and is called the error of the initial displacement.

The quantity $\delta_{k,\tau}$ is the result of round-off at each step (and also is due to the error of the integration method) and is called the round-off error.

Finally, the quantity $\delta_{k,q}$ is the result of omitting the remainder term in the integration formulas (14) and (17) and is called the quadrature error.

Substituting (16') in (33) and approximating the sum by an integral, we find the estimate for the quadrature error

$$|\delta_{k,q}| \leqslant \left| \alpha_{2\nu+2} h^{2\nu+2} \int_{t_0}^{t_k} V(\xi, t_k) \frac{d^{2\nu+2} f(\xi)}{dt^{2\nu+2}} d\xi \right| + \tag{34}$$
$$+ |\alpha_{2\nu+3}| (\nu+1) h^{2\nu+4} \int_{t_0}^{t_k} \left| V(\xi, t_k) \frac{d^{2\nu+4} f(\xi)}{dt^{2\nu+4}} \right| d\xi,$$

or disregarding, for sufficiently small h, the second term of (16')

$$\delta_{k,q} \approx -\alpha_{2\nu+2} h^{2\nu+2} \int_{t_0}^{t_k} V(\xi, t_k) \frac{d^{2\nu+2} f(\xi)}{dt^{2\nu+2}} \cdot d\xi. \tag{34'}$$

Usually in practice the quantities $\tau_k^{(i)} \equiv (\tau_k)^{(i)}$ are limited by the absolute value

$$|\tau_k^{(i)}| \leqslant \tau.$$

Taking absolute values of (32) and replacing all $|\tau_k^{(i)}|$ by τ, we obtain the estimate of the round-off error in the form of Lozinsky

$$|\delta_{k,\tau}^{(i)}| \leqslant \frac{\tau}{h^2} N_k^{(i)}, \tag{35}$$

where

$$N_k^{(i)} \equiv \int_{t_0}^{t_k} \sum_{j=1}^{n} |v_{ij}(\xi, t_k)| \, d\xi.$$

According to (26) and (26'), in the sum method τ takes the value

$$\tau \equiv \frac{4}{3}\rho + 4 \text{ er.}\left(\frac{1}{12}\right) M + 4\mu + \frac{1}{3}\lambda,$$

and in the case of the difference method, the value

$$\tau \equiv \frac{4}{3}\rho + 4 \text{ er.}\left(\frac{1}{12}\right) M + \mu' + \frac{1}{3}\lambda,$$

where it is assumed

$$M \equiv \max\left[h^2 \left| F^{(i)}(t_m, X_m) \right|\right] \approx \max\left[h^3 \left| F^{(i)}(t_m, \bar{x}_m) \right|\right],$$
$$1 \leqslant i \leqslant n, \quad 0 \leqslant m \leqslant k.$$

Estimate (35) can be improved a little if use is made of the following identity

$$\sum_{m=0}^{k-1} V(t_m, t_k)\,\Delta^2\delta_m \equiv V(t_0, t_k)\,\delta_{-1} + [V(t_1, t_k) - 2V(t_0, t_k)]\,\delta_0 +$$
$$+ \sum_{m=1}^{k-2} \Delta_m^2 V(t_m, t_k)\,\delta_m + [V(t_{k-2}, t_k) - 2V(t_{k-1}, t_k)]\,\delta_{k-1} + V(t_{k-1}, t_k)\,\delta_k, \qquad (*)$$

where δ_m are arbitrary vectors. Introducing the definition

$$V''(\xi, t) \equiv \frac{\partial^2 V(\xi, t)}{\partial \xi^2}$$

and utilizing the relations

$$\Delta_m^2 V(t_m, t_k) \equiv V(t_{m-1}, t_k) - 2V(t_m, t_k) + V(t_{m+1}, t_k) \approx h^2 V''(t_m, t_k),$$
$$V''(t_k, t_k) = 0, \quad V(t_{k-1}, t_k) \approx hI,$$

easily deduced from (3), (4), and (6), we shall rewrite the identity (*) in the form

$$\sum_{m=0}^{k-1} V(t_m, t_k)\,\Delta^2\delta_m \approx V(t_0, t_k)\,\delta_{-1} + [V(t_1, t_k) - 2V(t_0, t_k)]\,\delta_0 +$$
$$+ h^2 \sum_{m=1}^{k} V''(t_m, t_k)\,\delta_m + h\delta_k. \qquad (**)$$

We shall now substitute (26) in (32) and the result of the substitution we shall write in the form

$$\delta_{k,\tau} \equiv \delta_{k,\rho} + \delta_{k,\rho^*} + \delta_{k,\Phi} + \delta_{k,\mu} + \delta_{k,\lambda}, \tag{36}$$

where it is assumed

$$\delta_{k,\rho} \equiv \sum_{m=0}^{k-1} V(t_m, t_k) \frac{\rho_m}{h},$$
$$\delta_{k,\mu} \equiv \sum_{m=0}^{k-1} V(t_m, t_k) \Delta^2 \frac{\mu_m}{h}, \tag{37}$$
$$\cdots\cdots\cdots\cdots\cdots$$

Let us assume in the identity (**) $\theta_m = \mu_m$ and make with its help the estimate $|\delta^{(i)}_{k,\mu}|$, replacing all $|\mu^{(i)}_k|$ by their maximum value μ; then we shall obtain

$$|\delta^{(i)}_{k,\mu}| \leqslant \mu (1 + P^{(i)}_k),$$

where

$$P^{(i)}_k \equiv \int_{t_0}^{t_k} \sum_{j=1}^{n} |v''_{i,j}(\xi, t_k)| \, d\xi.$$

The estimate of the terms δ_{k,ρ^*}, $\delta_{k,\Phi}$, $\delta_{k,\lambda}$ from (36) is made analogously; thus it is revealed that (at least for equations of celestial mechanics) all these terms, differing from the term $\delta_{k,\rho}$ by the small factor h^2, represent small values in comparison with it, so that the estimate (35) for the sum method can be written in the form

$$|\delta^{(i)}_{k,\tau}| \approx |\delta^{(i)}_{k,\rho}| \leqslant \frac{\rho}{h^2} N^{(i)}_k. \tag{38}$$

As for the estimate of round-off error in Cowell's difference method, substituting (26') in (32) and reasoning as in the preceding case, one can show that here the principal terms will have the values $\delta_{k,\rho}$ and

$$\delta_{k,\mu'} \equiv \sum_{m=0}^{k-1} V(t_m, \; t_k) \frac{\mu'_m}{h}, \tag{37'}$$

and with respect to them, the terms δ_{k,ρ^*}, $\delta_{k,\Phi}$, $\delta_{k,\lambda}$ will give small corrections, so that the estimate (35) can be replaced by the following:

$$|\delta_{k,\tau}^{(i)}| \approx |\delta_{k,\rho}^{(i)} + \delta_{k,\mu'}^{(i)}| \leqslant \frac{\rho + \mu'}{h^2} N_k^{(i)}. \tag{38'}$$

Thus, knowing the fundamental solutions $U(t_0, t)$, $V(t_0, t)$ of the system $\ddot{y} = J(t) y$ and using the estimates (34') and (38) or (34') and (38'), we can, according to (30), estimate the error (28) in the integration of System (10) by Method (17) or (14):

$$|\delta_k| \leqslant |\delta_{k,0}| + |\delta_{k,\tau}| + |\delta_{k,\rho}|.$$

Remark. By integrating the equations of celestial mechanics always with a sufficiently large number of differences in the integration formulas and with sufficient accuracy in the initial conditions, so that in the right member of the above inequality the first two terms have practically no influence on the estimate of the error δ_k, then this inequality, taking account of (38) and (38'), can be written in the form

$$|\delta_k^{(i)}| \leqslant \frac{\rho}{h^2} N_k^{(i)} \tag{39}$$

(for the sum method) or

$$|\delta_k^{(i)}| \leqslant \frac{\rho + \mu'}{h^2} N_k^{(i)} \tag{39'}$$

(for the difference method) where the quantities $N_k^{(i)}$ are taken from (35).

The estimates (39) and (39') can be improved if, as adopted in celestial mechanics, round-off errors $\rho_k^{(i)}$, $\mu_k'^{(i)}$ are considered as independent random variables in the sense of the theory of probability (see the remark at the end of Paragraph 3).

5. ON CONDITIONS OF THE APPLICABILITY OF THE NORMAL DISTRIBUTION LAW OF PROBABILITIES TO THE ESTIMATION OF ROUND-OFF ERROR

Some real variable ξ defined by a random quantity is called stochastic (or simply "random variable"), if for any fixed x the probability of the inequality $\xi < x$, designated by $\varphi(x)$, is known; by varying x from $-\infty$ to $+\infty$ we obtain the function $\varphi(x)$, which is called the distribution law or the distribution function of the random variable ξ.

The relation of the random variable ξ to its distribution function $\varphi(x)$ is written in the following form:

$$P(\xi < x) = \varphi(x).$$

The argument of P can be transformed to equivalent inequalities without changing their probability.

Axioms of probability impose on the distribution function of any random variable certain limitations:

1) $\varphi(x)$ monotonically increases in the interval $-\infty < x < +\infty$;

2) $\varphi(-\infty) = 0$, $\varphi(+\infty) = 1$, etc.

Knowing the distribution function of the random variable ξ, for the given quantities x_1, x_2 one can find the probability of the inequality $x_1 < \xi < x_2$ by the formula

$$P(x_1 < \xi < x_2) = \varphi(x_2) - \varphi(x_1),$$

whence, in particular, it follows that

$$P(|\xi| < x) = \varphi(x) - \varphi(-x). \tag{40}$$

The random variable ζ, having the distribution function

$G(z) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}z^2} dz$, is called normal, and the function $G(z)$ is called the normal distribution law of probabilities. According to (40), for the normal random variable ζ we have

$$P(|\zeta| < z) = \Phi(z) \equiv \frac{2}{\sqrt{2\pi}} \int_{0}^{z} e^{-\frac{1}{2}z^2} dz. \tag{41}$$

We cite a few values of the function $\Phi(z)$:

z	$\Phi(z)$	z	$\Phi(z)$
0.03	0.0239	0.674	0.5000
0.1	0.0798	1.0	0.6826
0.2	0.1586	1.5	0.8650
0.3	0.2360	2.0	0.9544
0.4	0.3108	3.0	0.9973
0.5	0.3830	4.0	0.9999

(42)

For the random variable ξ, the distribution function of which has a continuous derivative $p(x)$, the following definitions are introduced:

1) expected value

$$E\xi \equiv \int_{-\infty}^{+\infty} xp(x)dx \equiv a;$$

2) variance

$$D(\xi) \equiv E(\xi - a)^2 \equiv b;$$

3) third absolute central moment

$$E|\xi - a|^3 \equiv c.$$

If λ is a constant quantity, then

$$E\lambda\xi = \lambda E\xi, \; D(\lambda, \xi) = \lambda^2 D(\xi).$$

The random variables $\xi_1, \xi_2, \ldots, \xi_k$ are called totally independent if the probability of the inequality $\xi_m < x$ for any fixed m is not dependent on the values the remaining random variables take.

THE CENTRAL LIMIT THEOREM OF THE THEORY OF PROBABILITY

Let $\xi_1, \xi_2, \ldots, \xi_k$ be some totally independent random variables, possessing the expected values $E\,\xi_m \equiv a_m$, the variances b_m and the third absolute central moments $c_m (1 \leqslant m \leqslant k)$.

Then the distribution law of the quantity

$$\zeta_k \equiv \frac{\sum_{m=1}^{k} (\xi_m - a_m)}{\sqrt{B_k}},$$

where $B_k \equiv \sum_{m=1}^{k} b_m$ tends to the normal law $G(z)$ for $k \to \infty$, if is satisfied the following condition of Lyapunov:

$$\psi_k \equiv \frac{\sum_{m=1}^{k} c_m}{B_k^{\frac{3}{2}}} \to 0 \text{ for } k \to \infty.$$

The theorem signifies, that

$$P\left(\frac{\sum_{m=1}^{k} (\xi_m - a_m)}{\sqrt{B_k}} < z\right) \approx G(z),$$

or, according to (41),

$$P\left(\left|\sum_{m=1}^{k} (\xi_m - a_m)\right| < z\sqrt{B_k}\right) \approx \Phi(z).$$

Returning now to the notation of Paragraph 4, we shall consider the round-off errors $\rho_m^{(j)}$ from (37) as independent random variables having the same distribution functions

$$\varphi(x)=0 \text{ for } x<-\rho,$$
$$\varphi(x)=\frac{x+\rho}{2\rho} \text{ for } -\rho\leqslant x\leqslant\rho,$$
$$\varphi(x)=1 \text{ for } x>\rho$$

(uniform distribution of a random variable).

The quantity ρ, appearing in the definition $\varphi(x)$, must satisfy the inequalities

$$|\rho_m^{(j)}|\leqslant\rho. \tag{43}$$

Applying the theorem to the random variable $\delta_{k,\rho}^{(i)}$, we find successively

$$E\rho_m^{(j)}=0,\quad D(\rho_m^{(j)})=\frac{\rho^2}{3},\quad 0\leqslant m\leqslant k-1;$$

$$B_k^{(i)}=\frac{\frac{1}{3}\rho^2}{h^2}\sum_{m=0}^{k-1}\sum_{j=1}^{n}v_{ij}^2(t_m, t_k)\approx\frac{\frac{1}{3}\rho^2}{h^3}N_k^{(i)},$$

where it is assumed

$$N_k^{(i)}\equiv\int_{t_0}^{t_k}\sum_{j=1}^{n}v_{ij}^2(\xi, t_k)\,d\xi. \tag{44}$$

The above theorem leads to the following statement. If the condition

$$\psi_k=\frac{\int_{t_0}^{t_k}\sum_{j=1}^{n}|v_{ij}^3(\xi, t_k)|\,d\xi}{(N_k^{(i)})^{\frac{3}{2}}}\to 0 \quad\text{for } k\to\infty, \tag{*}$$

is fulfilled, then to the round-off error $\delta_{k,\rho}^{(i)}$ we apply the normal distribution law of probabilities

$$P\left(|\delta_{k,\rho}^{(i)}| < z\sqrt{B_k^{(i)}}\right) = \Phi(z). \tag{45}$$

Remark. In all cases, which we subsequently encounter, the condition (*) is fulfilled, and we will not dwell on it further.

Let us introduce the coefficient of overestimating of Formula (45) $\eta = \frac{z\sqrt{B_k^{(i)}}}{|\delta_{k,\rho}^{(i)}|}$, which is itself a random variable

$$P(\eta < y) = P\left(\frac{z}{y}\sqrt{B_k^{(i)}} < |\delta_{k,\rho}^{(i)}|\right) = 1 - \Phi\left(\frac{z}{y}\right).$$

The quantity d, defined by the equation $d = P(\eta < 1) = 1 - \Phi(z)$, represents the probability of underestimating and must be considered as the defect of Formula (45) for a given z.

Of greatest interest is the quantity

$$p(y) = P(1 < \eta < y) = \Phi(z) - \Phi\left(\frac{z}{y}\right), \tag{46}$$

which characterizes the probability of "cleanly" overestimating given by Formula (45).

For $z = 3$ Formula (45) gives

$$|\delta_{k,\rho}^{(i)}| < \frac{\rho\sqrt{3}}{h^{\frac{3}{2}}}\sqrt{N_k^{(i)}} \tag{47}$$

with a probability of 0.9973, where the quantities ρ, $N_k^{(i)}$ are determined according to (43) and (44).

With regard for the observation made at the end of Paragraph 4, one can write the following probability estimate for the error of the sum method (17):

$$|\delta_k^{(i)}| < \frac{\rho\sqrt{3}}{h^{\frac{3}{2}}}\sqrt{N_k^{(i)}} = \varepsilon_k^{(i)}. \tag{48}$$

The probability of underestimating will be $d=0.0027$.

With the help of (42) and (46) one can determine the limits of "cleanly" overestimating, given by Formula (47) or (48) and the probabilities of their realization.

ν	$p(\nu)$
6	0.6143
10	0.7613
30	0.9175
100	0.9734
. . .	. . .

Thus, for example, according to the resulting data, one can assert that with a probability of 0.7613 Formula (47) gives a "clean" overestimate by less than a factor of 10.

Remark. Repeating the above discussion, we can write the probability estimate for the terms $\delta_{k,\mu}$, $\delta_{k,\rho}$ from (36) if the errors $\mu_m^{(i)}$, $\rho_m^{*(i)}$ are considered independent random variables and use is made of the identity (*) from Paragraph 4 [here are obtained small corrections to the right members of the inequalities (47)], and also the estimate of the term (37') if the errors $\mu_m'^{(i)}$ are considered independent quantities. Hence, making the estimate of (37') together with (37), we shall obtain the approximate estimate of round-off error for the difference method (14)

$$\left|\delta_{k,\rho}^{(i)} + \delta_{k,\mu'}^{(i)}\right| \leqslant \frac{\sqrt{\rho^2+\mu'^2}\,\sqrt{3}}{h^{\frac{3}{2}}} \cdot \sqrt{N_k^{(i)}},$$

which, according to the observation at the end of Paragraph 4, can be extended then to the error δ_k.

6. ESTIMATION OF THE ERROR OF NUMERICAL INTEGRATION OF THE PROBLEM OF TWO BODIES (ELLIPTICAL MOTION)

Let us consider the system of equations of undisturbed motion

$$\ddot{x} + \frac{K^2 x}{r^3} = 0, \quad \ddot{y} + \frac{K^2 y}{r^3} = 0, \quad \ddot{z} + \frac{K^2 z}{r^3} = 0, \qquad (49)$$
$$r^2 = x^2 + y^2 + z^2$$

and let us assume that it is integrated by the sum method. The general solution of System (49) has the form

$$\begin{aligned} x &= a\left[P_x(\cos E - e) + \sqrt{1-e^2}\,Q_x \sin E\right], \\ y &= a\left[P_y(\cos E - e) + \sqrt{1-e^2}\,Q_y \sin E\right], \\ z &= a\left[P_z(\cos E - e) + \sqrt{1-e^2}\,Q_z \sin E\right], \end{aligned} \qquad (50)$$

where

$$E - e\sin E = n(t-T), \quad n \equiv Ka^{-\frac{3}{2}}.$$

Here $P_x, P_y, P_z, Q_x, Q_y, Q_z$ denote direction cosines (Subbotin, 1941).

Let us introduce the vectors

$$\begin{gathered} \bar{x} \equiv \{x, y, z\} \equiv \{x^{(1)}, x^{(2)}, x^{(3)}\}, \\ \bar{x}^*(t) \equiv \{a(\cos E - e), \; a\sqrt{1-e^2}\sin E, \; 0\} \end{gathered} \tag{50'}$$

and the orthogonal matrix

$$S \equiv \| s_{ij} \| \equiv \begin{vmatrix} P_x, & Q_x, & \pm\sqrt{1-P_x^2-Q_x^2} \\ P_y, & Q_y, & \pm\sqrt{1-P_y^2-Q_y^2} \\ P_z, & Q_z, & \pm\sqrt{1-P_z^2-Q_z^2} \end{vmatrix}. \tag{51}$$

The signs in the third column of the matrix S are taken in such a way that the orthogonality condition is fulfilled, i.e.,

$$\tilde{S}S = I. \tag{52}$$

Then Formula (50) can be written in the form

$$\bar{x}(t) = S\bar{x}^*(t). \tag{53}$$

In the notation of Paragraph 4 we find

$$J(t, \bar{x}) = \frac{K^2}{r^5}\begin{vmatrix} 3x^2 - r^2, & 3xy, & 3xz \\ 3xy, & 3y^2 - r^2, & 3yz \\ 3xz, & 3yz, & 3z^2 - r^2 \end{vmatrix}.$$

Desiring to simplify the algorithm for computing the fundamental solutions $U(t_0, t)$, $V(t_0, t)$ of the system

$$\ddot{y} = J(t)\, y,$$

we introduce the matrix

$$J^*(t) \equiv J(t, \ \bar{x}^*(t))$$

and the fundamental solutions $U^*(t_0, t)$, $V^*(t_0, t)$ of the system

$$\ddot{y}^* = J^*(t)\,\bar{y}^*.$$

Then according to (1), (2), (52), and (53), we find successively

$$\begin{aligned} J(t) &= \frac{K^2}{r^5(t)}[3\bar{x}(t)\,\tilde{\bar{x}}(t) - r^2(t)\,I] = SJ^*(t)\tilde{S}; \\ U(t_0, t) &= SU^*(t_0, t)\,\tilde{S}, \\ V(t_0, t) &= SV^*(t_0, t)\,\tilde{S}. \end{aligned} \tag{54}$$

Proof of the relations (54) is immediately and sufficiently convincing by the fact that, using (53), their right members satisfy the conditions of Type (4).

Matrix $J^*(t)$ has the form

$$J^*(t) = \begin{vmatrix} A, & B, & 0 \\ B, & C, & 0 \\ 0, & 0, & D \end{vmatrix},$$

where

$$A \equiv n^2 \frac{(3 - e^2)\cos^2 E - 4e\cos E + (3e^2 - 1)}{(1 - e\cos E)^5},$$

$$B \equiv n^2 \frac{3\sqrt{1 - e^2}\,(\sin E\cos E - e\sin E)}{(1 - e\cos E)^5},$$

$$C \equiv n^2 \frac{(2e^2 - 3)\cos^2 E + 2e\cos E + (2 - 3e^2)}{(1 - e\cos E)^5},$$

$$D \equiv n^2 \frac{-1}{(1 - e\cos E)^3}.$$

Hence

$$\begin{aligned} v^*_{1,3}(t_0, t) &= 0, & v^*_{2,3}(t_0, t) &= 0, \\ v^*_{3,1}(t_0, t) &= 0, & v^*_{3,2}(t_0, t) &= 0, \end{aligned}$$

and the quantities $v^{\bullet}_{1,1}$, $v^{\bullet\bullet}_{1,2}$, $v^{\bullet}_{2,1}$, $v^{\bullet}_{2,2}$, $v^{\bullet}_{3,3}$ satisfy the equations

$$\begin{aligned} \ddot{v}^{\bullet}_{1,j} &= Av^{\bullet}_{1,j} + Bv^{\bullet}_{2,j} \\ \ddot{v}^{\bullet}_{2,j} &= Bv^{\bullet}_{1,j} + Cv^{\bullet}_{2,j} \quad (j = 1,\ 2). \\ \ddot{v}^{\bullet}_{3,3} &= Dv^{\bullet}_{3,3} \end{aligned}$$

The solution of these equations has the form

$$\begin{gathered} v^{\bullet}_{1,1}(t_0,\ t) = \frac{1}{n(1 - e\cos E)(1 - e\cos E_0)}\{[(1 - e^2)\sin E\cos E + e\sin E] \times \\ \times \cos^2 E_0 + [(e^2 - 1)\cos^2 E + (e^3 - e)\cos E + 2(1 - e^2)]\cos E_0 \sin E_0 + \\ + [(e - e^3)\sin E\cos E + 2(e^2 + 1)\sin E]\cos E_0 + [-e\cos^2 E - \\ - 2(e^2 + 1)\cos E + 5e]\sin E_0 + [-2(1 - e^2)\sin E\cos E - 5e\sin E] - \\ - 3\sin E\sin E_0(E - E_0)\}; \\ v^{\bullet}_{1,2}(t_0,\ t) = \frac{\sqrt{1 - e^2}}{n(1 - e\cos E)(1 - e\cos E_0)}\{[\cos^2 E + e\cos E - 2]\cos^2 E_0 + \\ + [\sin E\cos E + e\sin E]\sin E_0\cos E_0 + [-e\cos^2 E + 2\cos E - e]\cos E_0 + \\ + [-e\sin E\cos E + 2\sin E]\sin E_0 + [\cos^2 E + e\cos E - 2] + \\ + 3\sin E\cos E_0(E - E_0)\}; \\ v^{\bullet}_{2,1}(t_0,\ t) = \frac{\sqrt{1 - e^2}}{n(1 - e\cos E)(1 - e\cos E_0)}\{[-\cos^2 E + e\cos E - 1]\cos^2 E_0 + \\ + [-\sin E\cos E + e\sin E]\sin E_0\cos E_0 + [-e\cos^2 E - 2\cos E - e]\cos E_0 + \\ + [-e\sin E\cos E - 2\sin E]\sin E_0 + [2\cos^2 E + e\cos E + 2] + \\ + 3\cos E\sin E_0(E - E_0)\}; \\ v^{\bullet}_{2,2}(t_0,\ t) = \frac{1}{n(1 - e\cos E)(1 - e\cos E_0)}\{[\sin E\cos E - e\sin E]\cos^2 E_0 + \\ + [-\cos^2 E + (e^3 + e)\cos E - 1]\sin E_0\cos E_0 + [-(e^3 + e)\sin E\cos E + 2\sin E]\cos E_0 + \\ + [e\cos^2 E - 2\cos E + e]\sin E_0 + [\sin E\cos E - e\sin E] - 3(1 - e^2)\cos E\cos E_0(E - E_0)\}; \\ v^{\bullet}_{3,3}(t_0,t) = \frac{1}{n}[\sin E\cos E_0 + (-\cos E + e)\sin E_0 - e\sin E]. \end{gathered} \tag{55}$$

The correctness of Equations (55) is ascertained by direct substitution of their right members in the above equations and verification of the appropriate initial conditions [see (4)]. Here by E_0 is meant the initial value of the anomaly

$$E_0 - e\sin E_0 = n(t_0 - T).$$

Passing now to the definition of the matrix $U^*(t_0, t)$, we have according to (6)

$$
\begin{aligned}
u^*_{1,1}(t_0, t) &= \frac{1}{(1-e\cos E)(1-e\cos E_0)^3}\{3\sin E(\cos E_0 - e)(E-E_0)+\ldots\},\\
u^*_{1,2}(t_0, t) &= \frac{\sqrt{1-e^2}}{(1-e\cos E)(1-e\cos E_0)^3}\{3\sin E\sin E_0(E-E_0)+\ldots\},\\
u^*_{2,1}(t_0, t) &= \frac{\sqrt{1-e^2}}{(1-e\cos E)(1-e\cos E_0)^3}\{-3\cos E(\cos E_0 - e)(E-E_0)+\ldots\},\\
u^*_{2,2}(t_0, t) &= \frac{1}{(1-e\cos E)(1-e\cos E_0)^3}\{-3(1-e^2)\sin E_0\cos E(E-E_0)+\ldots\},\\
u^*_{3,3}(t_0, t) &= \ldots,
\end{aligned}
\qquad (5
$$

where the dots denote periodic terms, which for us have no importance.

The relations (54) give

$$
\begin{aligned}
v_{i,j}(t_0, t) = v^*_{1,1}(t_0, t)s_{i,1}s_{j,1} + v^*_{1,2}(t_0, t)s_{i,1}s_{j,2} + v^*_{2,1}(t_0, t)s_{i,2}s_{j,1} + \\
+ v^*_{2,2}(t_0, t)s_{i,2}s_{j,2} + v^*_{3,3}(t_0, t)s_{i,3}s_{j,3}
\end{aligned}
$$

and the analogous equation for $u_{i,j}(t_0, t)$, whence, introducing the quantities

$$
\begin{aligned}
\sigma^{(i)}_k &= s_{i,1}\sin E_k - \sqrt{1-e^2}\,s_{i,2}\cos E_k,\\
\gamma^{(i)}_k &= s_{i,1}(\cos E_k - e) + \sqrt{1-e^2}\,s_{i,2}\sin E_k \qquad (i=1,\ 2,\ 3),
\end{aligned}
$$

where

$$
E_k - e\sin E_k = n(t_k - T),
$$

according to (55) and (56) we find to within the secular terms

$$
\begin{aligned}
v_{i,j}(t_0, t_k) &= \frac{-3\sigma^{(i)}_k\sigma^{(j)}_0}{n(1-e\cos E_k)(1-e\cos E_0)}(E_k - E_0),\\
u_{i,j}(t_0, t_k) &= \frac{3\sigma^{(i)}_k\gamma^{(j)}_0}{(1-e\cos E_k)(1-e\cos E_0)^3}(E_k - E_0).
\end{aligned}
\tag{57}
$$

Substituting (57) in (31) we obtain the expression for the error of the initial displacement

$$
\begin{aligned}
\delta^{(i)}_{k,0} \approx \frac{3\sigma^{(i)}_k}{(1-e\cos E_k)(1-e\cos E_0)^3}\left|\gamma^{(1)}_0\delta^{(1)}_0 + \gamma^{(2)}_0\delta^{(2)}_0 + \gamma^{(3)}_0\delta^{(3)}_0\right|(E_k - E_0) + \\
+ \frac{-3\sigma^{(i)}_k}{nh(1-e\cos E_k)(1-e\cos E_0)}\left|\sigma^{(1)}_0(\delta^{(1)}_0 - \delta^{(1)}_{-1}) + \sigma^{(2)}_0(\delta^{(2)}_0 - \delta^{(2)}_{-1}) + \sigma^{(3)}_0(\delta^{(3)}_0 - \delta^{(3)}_{-1})\right|(E_k - E_0).
\end{aligned}
$$

Equation (34') with regard for the relations (53) and (54) gives the estimate for the quadrature error

$$\bar{\delta}_{k,q} \approx -\alpha_{2\nu+2} S \int_{t_0}^{t_k} V^*(\xi, t_k) \frac{d^{2\nu+4}\bar{x}^*(\xi)}{dt^{2\nu+4}} d\xi.$$

Hence with the aid of (50') and (55) we find to within the secular terms (for simplicity the formulas are written for $e = 0$)

$$\delta^{(i)}_{k,q} \approx \pm 2\alpha_{2\nu+2}\sigma^{(i)}_k (nh)^{2\nu+2}(E_k - E_0).$$

We turn now to the calculation of the quantities $N^{(i)}_k$, defined by Equations (44) and appearing in the estimate of the round-off error (47). For this we notice, that these quantities form the main diagonal of the matrix $V\tilde{V}$, which, according to (54), can be represented in the form

$$V\tilde{V} = SV^*\tilde{V}^*\tilde{S}.$$

Hence, introducing the notation

$$N^{*(i,j)}_k = \int_{t_0}^{t_k} \left[v^*_{i,1}(\xi, t_k)v^*_{j,1}(\xi, t_k) + v^*_{i,2}(\xi, t_k)v^*_{j,2}(\xi, t_k)\right] d\xi \qquad (i = 1, 2, \quad j = 1, 2),$$

$$N^{*(3,3)}_k = \int_{t_0}^{t_k} v^{*2}_{3,3}(\xi, t_k)\, d\xi,$$

we find

$$N^{(i)}_k = N^{*(1,1)}_k s^2_{i,1} + 2N^{*(1,2)}_k s_{i,1}s_{i,2} + N^{*(2,2)}_k s^2_{i,2} + N^{*(3,3)}_k s^2_{i,3} \quad (i = 1, 2, 3). \qquad (58)$$

Elementary calculations according to (55) lead to the following expressions for the quantities $N_k^{*(i,j)}$ (which for simplicity were written for $e=0$):

$$N_k^{*(1,1)} = \frac{1}{n^3}\Big\{3\sin^2 E_k (E_k - E_0)^3 + 6\sin E_k \cos E_k (E_k - E_0)^2 +$$
$$+\Big[\frac{13}{2} + \frac{15}{2}\sin^2 E_k + 12\sin^2 E_k \cos(E_k - E_0) + 12\sin E_k \sin E_0\Big](E_k - E_0) +$$
$$+\Big[24\cos^2 E_k \sin(E_k - E_0) - 32\sin(E_k - E_0) + 3(\sin E_k \cos E_k - \sin E_0 \cos E_0) -$$
$$-\frac{3}{4}(\sin^2 E_k + 1)\sin 2(E_k - E_0)\Big]\Big\},$$
$$N_k^{*(1,2)} = \frac{1}{n^3}\Big\{-3\sin E_k \cos E_k (E_k - E_0)^3 - 3(\cos^2 E_k - \sin^2 E_k)(E_k - E_0)^2 +$$
$$+\Big[-\frac{15}{2}\sin E_k \cos E_k - 12\sin E_k \cos E_k \cos(E_k - E_0) - 6(\sin E_k \cos E_0 + \sin E_0 \cos E_k)\Big] \times$$
$$\times (E_k - E_0) + \Big[24\sin E_k \cos E_k \sin(E_k - E_0) - \frac{3}{2}(\cos^2 E_k - \sin^2 E_k) +$$
$$+\frac{3}{2}(\cos^2 E_0 - \sin^2 E_0) + \frac{3}{4}\sin E_k \cos E_k \sin 2(E_k - E_0)\Big]\Big\},$$
$$N_k^{*(2,2)} = \frac{1}{n^3}\Big\{3\cos^2 E_k (E_k - E_0)^3 - 6\sin E_k \cos E_k (E_k - E_0)^2 +$$
$$+\Big[\frac{13}{2} + \frac{15}{2}\cos^2 E_k + 12\cos^2 E_k \cos(E_k - E_0) + 12\cos E_k \cos E_0\Big](E_k - E_0) +$$
$$+\Big[24\sin^2 E_k \sin(E_k - E_0) - 32\sin(E_k - E_0) - 3(\sin E_k \cos E_k - \sin E_0 \cos E_0) -$$
$$-\frac{3}{4}(\cos^2 E_k + 1)\sin 2(E_k - E_0)\Big]\Big\},$$
$$N_k^{*(3,3)} = \frac{1}{n^3}\Big\{\frac{1}{2}(E_k - E_0) - \frac{1}{2}\sin(E_k - E_0)\cos(E_k - E_0)\Big\}.$$

Hence, having used the quantities

$$\begin{aligned}\sigma_k^{(i)} &= s_{i,1}\sin E_k - s_{i,2}\cos E_k \\ \gamma_k^{(i)} &= s_{i,1}\cos E_k + s_{i,2}\sin E_k\end{aligned} \qquad (i = 1,\ 2,\ 3),$$

from (58) we find

$$N_k^{(i)} = \frac{1}{n^3}\Big\{3\sigma_k^{(i)2}(E_k - E_0)^3 + 6\sigma_k^{(i)}\gamma_k^{(i)}(E_k - E_0)^2 + \Big[\frac{13}{2} - 6s_{i,3}^2 + \frac{15}{2}\sigma_k^{(i)2} + 12\sigma_k^{(i)2}\cos(E_k - E_0) +$$
$$+ 12\sigma_k^{(i)}\sigma_0^{(i)}\Big](E_k - E_0) + \Big[-8(1 - s_{i,3}^2)\sin(E_k - E_0) - 24\sigma_k^{(i)2}\sin(E_k - E_0) + \tag{59}$$
$$+ 3(\sigma_k^{(i)}\gamma_k^{(i)} - \sigma_0^{(i)}\gamma_0^{(i)}) + \Big(-\frac{3}{4}\sigma_k^{(i)2} - \frac{3}{4} + \frac{1}{2}s_{i,3}^2\Big)\sin 2(E_k - E_0)\Big]\Big\} \qquad (i = 1,\ 2,\ 3).$$

For $e \neq 0$, Equations (59) must be replaced by the following (for $s_{i,3} = 0$):

$$
\begin{aligned}
N_k^{(i)} = \frac{1}{n^3 (1 - e\cos E_k)^3} \Big\{ & 3\sigma_k^{(i)2} (E_k - E_0)^3 + \left[6\sigma_k^{(i)}\gamma_k^{(i)} - 3e\sin E_k \sigma_k^{(i)2} - 9e \sin E_0 \sigma_k^{(i)2}\right] (E_k - E_0)^2 + \\
& + \frac{1}{1-e^2} \Big[\frac{3}{2} e^2 (1 - e^2) \sigma_k^{(i)2} \cos 2E_0 + (1 - e^2) [24 (\cos E_k + e) \sigma_k^{(i)2} - \\
& - 12 \sin E_k \sigma_k^{(i)} \gamma_k^{(i)}] \cos E_0 + (1 - e^2) [6 (e^2 + 4) \sin E_k \sigma_k^{(i)2} + \\
& + 12 \cos E_k \sigma_k^{(i)} \gamma_k^{(i)}] \sin E_0 + \left[2e^2 \cos^2 E_k + 5e\cos E_k + \left(\frac{13}{2} - \frac{7}{2} e^2\right)\right] \gamma_k^{(i)2} + \\
& + \left[(e^4 - 9e^2)\cos^2 E_k + (4e - 8e^3)\cos E_k + \left(-\frac{3}{2} e^4 - \frac{1}{2} e^2 + 14\right)\right] \sigma_k^{(i)2} + \\
& + [8e^2 \sin E_k \cos E_k + (8e^3 + 4e) \sin E_k] \sigma_k^{(i)} \gamma_k^{(i)} \Big] (E_k - E_0) + [\ldots] \Big\},
\end{aligned} \tag{60}
$$

where

$$
\begin{aligned}
\sigma_k^{(i)} &\equiv s_{i,1} \sin E_k - \sqrt{1 - e^2}\, s_{i,2} \cos E_k, \\
\gamma_k^{(i)} &\equiv [(1 + e^2) \cos E_k - 2e]\, s_{i,1} + \sqrt{1 - e^2}\, s_{i,2} \sin E_k.
\end{aligned}
$$

Here the dots denote periodic terms differing by a small quantity from corresponding terms of (59).

Inversion of Keppler's equation [see (50)], gives

$$E = M + e \sin M + \frac{1}{2} e^2 \sin 2M + \ldots,$$

where

$$M \equiv n (t - T);$$

hence

$$E_k - E_0 = knh + 2e \sin \frac{knh}{2} \cos \frac{M_0 + M_k}{2} + e^2 \sin knh \cos(M_0 + M_k) + \ldots$$

If k corresponds to an integral number of revolutions, then

$$E_k - E_0 = knh.$$

For a sufficiently large value of knh and for $\sigma_k^{(i)} \neq 0$ Formulas (47) and (60) give the following approximate expression for the round-off error:

$$|\delta_{k,\rho}^{(i)}| < \varepsilon_k^{(i)} \approx \frac{3\rho\,|\sigma_k^{(i)}|}{(1 - e\cos E_k)}\, k^{\frac{3}{2}}, \tag{61}$$

where

$$\sigma_k^{(i)} \equiv s_{i,1} \sin E_k - \sqrt{1 - e^2}\, s_{i,2} \cos E_k.$$

One can compare this result with the well known estimation of Newcomb, which in our notation can be written in the form

$$\|\bar{\delta}_{k,\rho}\| < 0.225\rho k^{\frac{3}{2}}.$$

From the standpoint of the theory, stated in Paragraph 5, the probability of such an inequality will be equal to 0.1780 [see (42) and (46)].

If use is made of the estimate of round-off error in the form of (35), then we obtain the following results:

$$|\delta_{k,\rho}^{(i)}| < \varepsilon_k'^{(i)} \approx \frac{3}{2}\sqrt{3}\rho\,|\sigma_k^{(i)}|\, k^2. \tag{62}$$

Remark 1. Let us consider the case of circular motion described by the equations

$$\ddot{x}^{(i)} + n^2 x^{(i)} = 0 \qquad (i = 1, 2, 3),$$

$$\sum_{i,1}^{3} x^{(i)2} = r^2 = \text{const},$$

where n^2 is a constant.

Let us suppose that this system is integrated by Cowell's method. The general solution of the system has the form

$$x^{(i)} = r\,(s_{i,1}\cos nt + s_{i,2}\sin nt) \qquad (i = 1, 2, 3).$$

Calculating the estimate of the error of numerical integration of this system by Formulas (47), we find successively

$$v_{i,i}(t_0, t) = \frac{\sin n\,(t - t_0)}{n}, \qquad v_{i,j}(t_0, t) \equiv 0, \qquad i \neq j;$$

$$|\delta_{k,\rho}^{(i)}| \leqslant \varepsilon_k^{(i)} \equiv \frac{\sqrt{\frac{3}{2}}\rho}{(nh)^{\frac{3}{2}}} \sqrt{knh - \frac{1}{2}\sin 2knh} \approx \frac{\sqrt{\frac{3}{2}}\rho}{nh} k^{\frac{1}{2}}.$$

This result was predicted by Brouwer.

Remark 2. Let us consider at the same time the equations of disturbed and undisturbed motions

$$x_0 + \frac{K^2 x_0}{r_0^3} = R,$$

$$x + \frac{K^2 x}{r^3} = 0,$$

where R is the disturbing function. We shall denote respectively by $\bar{X}_0$, $\bar{X}$ the results of integrating these systems by Cowell's method with the same initial conditions. Then because of the small value of the disturbing function R one can put $\bar{X}_0 - x_0 \approx \bar{X} - x$ and estimate the error of numerical integration of the system of disturbed motion by the formulas of Paragraph 6. (This hypothesis is a supposition of S. M. Lozinsky.)

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THE ACCUMULATION OF NUMERICAL INTEGRATION ERRORS IN SOME PROBLEMS OF CELESTIAL MECHANICS

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The results of the application of estimations of errors in numerical integration (Myachin, 1959) to numerical examples are presented.

Recently in connection with the rapid development of computing equipment, methods of numerical integration for the solution of the problem of n bodies have become most effective. Such enormous works of calculation as "Coordinates of the Five Outer Planets 1653 - 2060", "Coordinates of Four Minor Planets 1940 - 1960", and others, were accomplished by methods of numerical integration. However, the accumulation of errors in integration essentially reduces the quality of the numerical methods.

Errors arise due to limited precision of the calculating machine (round-off errors), neglected differences in the integration formulas, and inaccurate values of the initial conditions.

Errors due to neglected differences can be reduced to a minimum by a suitable choice of the interval and the number of terms in the integration formula. The calculation of errors in the initial conditions does not present great difficulty: if the initial conditions are computed with the precision with which the computations are carried out, then one can consider these errors as round-off errors in the first step of integration.

In the majority of cases it is important to know the dependence between the number of steps and the number of lost digits, in order to provide for the necessary precision in advance or determine with what error any or all quantities were obtained.

With this aim B. F. Myachin (1959) investigated the formulas for estimation of the accumulation of round-off errors in numerical integration of the equations of motion of the two-body problem.

If the eccentricity, e, is neglected and the true error at the kth step is indicated by $\delta_k^{(i)}$, we can write these formulas in the following form:

$$|\delta_k^{(i)}| < \varepsilon_k^{(i)} \quad (k = 1, 2, 3\ldots; \; i = 1, 2, 3),$$

where k is the number of steps, i is the number of the coordinate (x, y, z),

$$\varepsilon_k^{(i)} = \frac{\rho\sqrt{3}}{(nh)^{3/2}} \sqrt{N^{(i)}(E_k)},$$

$$\begin{aligned} N^{(i)} = {} & 3\sigma_k^{(i)2}(E_k - E_0)^3 + 6\sigma_k^{(i)}\gamma_k^{(i)}(E_k - E_0)^2 + \left[\frac{13}{2} - 6s_{i,3}^2 + \frac{15}{2}\sigma_k^{(i)2} + 12\sigma_k^{(i)2}\cos(E_k - E_0) + \right. \\ & \left. + 12\sigma_k^{(i)}\sigma_0^{(i)}\right](E_k - E_0) + \left[-8(1 - s_{i,3}^2)\sin(E_k - E_0) - 24\sigma_k^{(i)2}\sin(E_k - E_0) + \right. \\ & \left. + 3(\sigma_k^{(i)}\gamma_k^{(i)} - \sigma_0^{(i)}\gamma_0^{(i)}) + \left(-\frac{3}{4}\sigma_k^{(i)2} - \frac{3}{4} + \frac{1}{2}s_{i,3}^2\right)\sin 2(E_k - E_0)\right], \end{aligned} \qquad (1)$$

$$\sigma_k^{(i)} = s_{i,1}\sin E_k - s_{i,2}\cos E_k,$$

$$\gamma_k^{(i)} = s_{i,1}\cos E_k + s_{i,2}\sin E_k,$$

$s_{i,1}$, $s_{i,2}$, $s_{i,3}$ are the direction cosines (in general denoted by P, Q, R), n is the diurnal motion of a body, h is the integration step, E_k is the eccentric anomaly (in the given case the mean, since $e = 0$), and ρ is the maximum round-off error in the computation of the right members of the equations at each step.

We used the formulas in this form for numerically comparing the error $\delta_k^{(i)}$, obtained by numerical integration, with the predicted $\varepsilon_k^{(i)}$.

1. We shall carry out the indicated comparison in the exsmples, which were specially computed for this purpose (the examples were computed on the electronic machine BESM AN-SSSR). The problem of plane undisturbed motion is solved with various initial conditions

(orbital elements). The value of the interval of integration and the elements are selected in such a way, that one revolution is accomplished in exactly 100 steps. In all, three examples, each having 1100 steps, were computed; the initial conditions were defined by the following elements:

	1st example	2nd example	3rd example
M_0	$0°$	$0°$	$0°$
ω	$0°$	$0°$	$0°$
e	0.04825380	0.04825380	0.2
n	$299''.128376$	$648''$	$648''$
Integration step h .	$43^d.35258794$	$20^d.0$	$20^d.0$

Integration was conducted by Cowell's sum method taking account of fourth differences

$$X = f^{(-2)} + \frac{1}{12} f - \frac{1}{240} f^{(2)} + \frac{31}{60480} f^{(4)},$$

where

$$f = -h^2 k^2 \frac{X}{r^3},$$

and X is a vector with the components X, Y, Z.

The results of the integration

$$X_1,\ X_2,\ \ldots,\ X_k,\ \ldots$$

were compared with the values

$$x_1,\ x_2,\ \ldots,\ x_k,\ \ldots,$$

computed beforehand by the formula of elliptic motion to the 6th decimal place, and the difference

$$x_k - X_k = \delta_k$$

was taken as the net accumulation of round-off errors, since the influence of the higher differences in the integration formula, in the given case $f^{(6)}$, was computed with the maximum precision $\left(\frac{317}{22809600} f^{(6)} < 1 \cdot 10^{-11}\right)$.

In the future we will not take these errors into account, since in all of the works considered their influence lies beyond the limit of precision with which the computations are performed.

For the round-off error, ρ, the adopted value is 0.5×10^{-9}; that is, the precision with which f was calculated at each step. One can note that Formulas (1) are considerably simplified if the estimates are computed for the points $E_k - E_0$ as multiples of $0, \frac{\pi}{2}, \pi, \frac{3}{2}\pi$. In our case, since the problem of plane motion was solved $(s_{11} = s_{22} = 1,\ s_{12} = s_{21} = 0)$ and integration began from the point of perigee $(E_0 = 0)$, Formulas (1) are simplified still further and become

$$|\delta_k^{(i)}| < \varepsilon_k^{(i)} = \frac{\sqrt{3}}{(nh)^{\frac{3}{2}}} \rho \sqrt{N^{(i)}(E_k)} \qquad (i = 1,\ 2;\ k = 1,\ 2,\ 3 \dots),$$

$$N^{(1)}(E_k) = \frac{13}{2} E_k, \quad N^{(2)}(E_k) = 3E_k^3 + 38E_k \quad \text{for } E_k = 2m\pi,\ (m = 1, 2 \dots)$$

$$N^{(1)}(E_k) = 3E_k^3 + 14E_k - 32, \quad N^{(2)}(E_k) = \frac{13}{2} E_k - 8 \quad \text{for } E_k = \left(2m + \frac{1}{2}\right)\pi, \tag{2}$$

$$N^{(1)}(E_k) = \frac{13}{2} E_k, \quad N^{(2)}(E_k) = 3E_k^3 - 10E_k \quad \text{for } E_k = (2m + 1)\pi,$$

$$N^{(1)}(E_k) = 3E_k^3 + 14E_k + 32, \quad N^{(2)}(E_k) = \frac{13}{2} E_k + 8 \quad \text{for } E_k = \left(2m + \frac{3}{2}\right)\pi.$$

Beginning with $k > 200$, the ratio $\frac{E_k}{E_k^3}$ is of the order of $\frac{1}{100}$; therefore, neglecting the first power of E_k in comparison with E_k^3 and letting $E_k = nhk$, we shall obtain

$$\begin{aligned} &\varepsilon_k^{(1)} = 70.3\rho k^{\frac{1}{2}}, \quad \varepsilon_k^{(2)} = 3\rho k^{\frac{3}{2}} \quad \text{for } E_k = m\pi, \\ &\varepsilon_k^{(1)} = 3\rho k^{\frac{3}{2}}, \quad \varepsilon_k^{(2)} = 70.3\rho k^{\frac{1}{2}} \quad \text{for } E_k = \left(m + \frac{1}{2}\right)\pi. \end{aligned} \tag{3}$$

We note that for all three of the examples under consideration $\varepsilon_k^{(i)}$ are the same. This is stipulated by the fact that the product nh in all the examples is one and the same $\left(\frac{\pi}{50}\right)$ and, in addition, the influence of eccentricity is excluded in the cited formulas.

However, the eccentricity does not very strongly distort the result. For $k > 200$ the formula of estimation (Myachin, 1959) is written in the form

$$\varepsilon_k = 3\rho \, |\sigma_k| \, \frac{k^{\frac{3}{2}}}{1 - e \cos E_k}.$$

In Table 1 are cited the computed $|\delta_k^{(i)}|$ and the predicted errors $\varepsilon_k^{(i)}$. The first column gives the mean anomaly for all three "planets"; the 2nd, the appropriate number of integration steps; the 3rd to 8th columns, the true errors $\delta_k^{(1)}$ and $\delta_k^{(2)}$ for examples I to III; the 9th and 10th columns, the values of $\varepsilon_k^{(1)}$ and $\varepsilon_k^{(2)}$, the estimate of the error in all three examples. The values $\delta_k^{(i)}$, $\varepsilon_k^{(i)}$ are expressed in units of the sixth decimal place.

Comparing the 3rd, 5th and 7th columns with the 9th; or the 4th, 6th, and 8th, with the 10th; we can judge the quality of the obtained estimate. Note that the estimate reflects the oscillatory character of the accumulation of errors.

2. From the standpoint of the accumulation of errors it turned out to be of interest to examine the coordinates of Uranus, Saturn, and Jupiter obtained by D. K. Kulikov by integrating the VIIIth satellite of Jupiter for the period from 24 January 1930 to 28 August 1965. The integration was carried out on the electronic calculating machine BESM with steps of 10 days (all 1300 steps). The coordinates of the planets were obtained by simultaneous integration of the system of nine equations; the initial coordinates were taken from "Astronomical Papers" (1951).

TABLE 1

Anomaly	Number of Steps	1st Example		2nd Example		3rd Example		Estimate	
		$\delta_k^{(1)}$	$\delta_k^{(2)}$	$\delta_k^{(1)}$	$\delta_k^{(2)}$	$\delta_k^{(1)}$	$\delta_k^{(2)}$	$\epsilon_k^{(1)}$	$\epsilon_k^{(2)}$
0	1	0	0	0	0	0	0	0	0
90	26	0	0	0	0	0	0	0	0
180	51	1	1	0	0	0	0	0	0
270	76	3	0	0	0	0	0	0	0
360	101	0	5	0	0	0	+1	0	1.5
450	126	3	2	0	0	0	0	2	0
540	151	0	3	0	1	1	3	0	3
630	176	6	0	0	0	0	0	4	0
720	201	0	7	0	0	0	1	0	4
810	226	6	0	+1	0	1	0	5	0
900	251	0	5	0	0	0	5	0	6
990	276	7	0	0	0	0	0	7	0
1080	301	0	9	0	0	0	0	1	8
1170	326	7	0	0	-1	+3	0	9	1
1260	351	0	5	0	-2	0	-6	1	10
1350	376	-7	-1	1	-1	-2	-3	11	1
1440	401	0	-11	0	-1	0	-1	1	12
1530	426	8	-1	-1	-1	-1	2	13	2
1620	451	1	5	0	-2	0	4	1	14
1710	476	-8	-1	1	-1	-2	-3	15	2
1800	501	0	-10	0	-1	0	-8	1	16
1890	526	6	-2	-1	-1	3	1	18	2
1980	551	0	3	0	-2	0	-2	2	19
2070	576	-6	-1	0	-1	-7	-1	21	0
2160	601	0	7	0	-1	0	-14	2	22
2250	626	1	-2	-1	1	7	1	24	2
2340	651	-1	-1	0	-1	0	-2	2	25
2340	676	-1	-1	-1	+1	-10	-1	26	2
2520	701	0	-3	0	-3	0	-21	2	27
2610	726	-2	-3	2	-1	12	-1	28	2
2700	751	2	-7	0	1	0	5	2	30
2790	776	4	-2	-4	-1	-15	-1	31	2
2880	801	0	2	0	-7	0	-29	2	33
2970	826	-6	-4	6	-1	16	-1	35	2
3060	851	2	-11	0	4	0	8	3	36
3150	876	9	-2	-8	-1	-20	0	38	3
3240	901	0	7	0	-12	0	-35	3	40
3330	926	-11	-3	9	-1	20	-1	42	3
3420	951	-2	-16	0	9	0	12	3	44
3510	976	14	0	-13	-1	-25	1	46	3
3600	1001	0	12	0	-19	0	-44	3	48
3690	1226	-15	-4	18	-1	36	-1	50	3
3780	1051	-2	-21	0	15	0	16	3	52
3870	1076	19	-2	-20	0	-31	2	53	3
3960	1101	0	17	0	-26	0	-53	3	54

Since the coordinates of the planets published in "Astronomical Papers" in 1951 were computed with great precision (the joint system of equations of motion of the five outer planets was solved; moreover, the calculations were carried out with 14 digits), they were taken for the precise solution x_k, and the difference $x_k - \overline{X}_k$ was taken for the true error of the estimated coordinates. The comparison is made in the 5th decimal place. The results are given in Table 2.

For an estimate of this error by the formulas of B. F. Myachin, it is necessary to ascertain the error made in a single step. In the given case, in addition to the round-off error $\rho (\rho = 0.5 \cdot 10^{-n})$, the computation of the right members of the equations will contain an error due to the neglected disturbances from Neptune and Pluto. The magnitude of the disturbances amounts to $1 \cdot 10^{-n}$; that is, it exceeds the computing error. Of the three planets, Uranus is subject to the greatest disturbing influences; since during the investigated interval of time Uranus makes only a half revolution (and the sum of the disturbances from Neptune and Pluto has a constant sign during a considerable period of time), while Saturn makes 1.5 revolutions and Jupiter 3.5; and accordingly the disturbances from Neptune and Pluto have a periodic character. For the estimate of the error due to round-off we use Formulas (1) (the results are given in Table 2 in columns 4, 7, 10). For the estimate of the error due to neglected disturbances, however, it is impossible to use these formulas since in deducing (1) the probability law of the distribution of random errors was used and the disturbances do not obey the law of random errors. The only acceptable formula in this case is the following (Myachin, 1959):

$$|\delta_k^{(i)}| < \varepsilon_k^{(i)} \approx \frac{3}{2} \sqrt{3}\, \rho\, |\sigma_k^{(i)}|\, k^2 \qquad (i = 1, 2, 3;\ k = 1, 2, 3 \ldots), \tag{4}$$

TABLE 2

Anomaly E_k	Number of steps	$x_k - X_k$	Round off error	Error due to neglected disturbances	$y_k - Y_k$	Round off error	Error due to neglected disturbances	$z_k - Z_k$	Round off error	Error due to neglected disturbances
					Jupiter					
135	90	0	0.1	1	0	0.1		0	0.0	
180	144	0	0.2	1	0	0.3	5	0	0.1	2
225	198	−2	0.4		0	0.1		0	0.8	
270	252	−3	0.5	15	−1	0.3	3	0	0.1	1
315	306	−2	0.3		−3	0.8		−1	0.3	
360	360	+1	0.5	7	−3	1.1	31	−1	0.5	16
405	414	+2	1.3		−2	0.6		0	0.3	
450	468	+3	1.6	56	0	0.6	11	0	0.5	4
495	522	+1	0.9		2	1.5		1	0.6	
540	576	−2	0.7	16	2	1.8	77	0	0.8	53
585	630	−3	2.1		2	1.0		0	0.5	
630	684	−6	2.6	120	−1	0.8	24	0	0.3	8
675	738	−4	1.4		−6	2.5		−2	1.1	
720	792	0	1.1	32	−8	3.0	147	−4	1.3	63
765	846	5	3.4		−4	1.6		−2	0.8	
810	900	5	4.0	207	0	1.2	42	0	0.4	13
855	954	2	2.2		4	3.8		2	1.5	
900	1008	−1	1.4	51	4	4.3	239	2	1.9	103
945	1066	−5	4.6		2	0.7		1	0.2	
990	1120	−8	5.4	319	−2	1.5	65	−1	0.5	20
1035	1174	−5	3.0		−10	5.0		−4	2.0	
1080	1228	+4	4.5	76	−10	5.0	353	−4	2.5	152
1125	1272	+8	6.1		−6	3.6		−3	2.2	
					Saturn					
225	120	+1	0.2		0	0.4		0	0.2	
270	256	+3	0.6	0.6	−1	1.0	16	0	0.4	6
315	392	+9	1.4		−1	1.1		−1	0.5	
360	528	+19	1.4	72	+6	0.9	4	+1	0.4	1.4
405	664	+16	1.2		+20	2.4		+7	1.0	
450	800	+1	1.2	6.4	+22	3.4	154	+9	1.4	64
495	936	−9	4.0		+12	2.6		+5	1.2	
540	1072	−17	6.2	299	−9	1.4	15	−4	0.6	6
					Uranus					
225	213	+2	3.3	7	−1	4.1	8	−1	2.0	4
270	587	+14	2.5	60	−7	1.5	12	−3	1.0	7
315	970	+31	5.2	188	−23	4.8	150	−11	4.8	47

where by ρ we denote the error in a single step due to the neglected disturbances, and $\delta_k^{(i)}$ has the same meaning as in (1).

In the computation by Formulas (4) for all three planets ρ is taken as the maximum disturbance from Neptune and Pluto; that is, $1 \cdot 10^{-9}$, which, of course, gives a strong overestimate for Jupiter and Saturn (Table 2, columns 5, 8, 11).

3. We shall give one more example. Let us try to estimate the error due to round-off in the coordinates of the larger planets published in "Astronomical Papers" in 1951. We shall make the estimate by the crude formula obtained from (1) under the following assumptions: we neglect E_k^2 and E_k in comparison with E_k^3; σ_k^i we assume equal to 1. Then (1) will have the form

$$|\delta_k^i| < \varepsilon_k^i \approx 3\rho k^{\frac{3}{2}}. \tag{5}$$

If ρ is taken to be $1 \cdot 10^{-14}$, then after 1000 steps of integration, which corresponds to an interval of time greater than 100 years in this case, the error in the coordinates of the planets will be approximately equal to $1 \cdot 10^{-9}$; that is, the published coordinates of the larger planets are free from round-off error.

Thus, the examples cited above show that the formulas derived by B. F. Myachin for the estimate of errors due to round-off errors is entirely suitable for practical use.

The estimate (1) reflects the oscillatory character of the error and gives a comparatively smaller overestimate (as a rule, by less than a factor of 10). Furthermore, it shows that after 1000 steps of integration no more than five digits are lost in the sought for values due to round-off errors.

But as for the error created by the neglected disturbances (Table 2), the estimate (4) which was used for them must be considered unsatisfactory, since it does not take into account the periodic character of the disturbances. This estimate gives a practically acceptable result only if during the entire integration or a large part of it the disturbances are constant.

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